LIPSCHITZ CONDITIONS AND THE DISTANCE RATIO METRIC

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Abstract. We give study the Lipschitz continuity of Möbius transformations of a punctured disk onto another punctured disk with respect to the distance ratio metric.

1. Introduction

During the past thirty years the theory of quasiconformal maps has been studied in various contexts such as in Euclidean, Banach, or even metric spaces. It has turned out that while some classical tools based on conformal invariants, real analysis and measure theory are no longer useful beyond the Euclidean context, the notion of a metric space and related notions still provide a useful conceptual framework. This has led to the study of the geometry defined by various metrics and to the key role of metrics in recent theory of quasiconformality. See e.g. [CCQ, HIMPS, HPWW, K, RT1, RT2, V].

Distance ratio metric. One of these metrics is the distance ratio metric. For a subdomain $G \subset \mathbb{R}^n$ and for $x, y \in G$ the *distance ratio metric* j_G is defined by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d_G(x), d_G(y)\}}\right), \tag{1.1}$$

where $d_G(x)$ denotes the Euclidean distance from x to ∂G. If $G_1 \subset G$ is a proper subdomain then for $x, y \in G_1$ clearly

$$j_G(x,y) \le j_{G_1}(x,y)\,. \tag{1.2}$$

Moreover, the numerical value of the metric is highly sensitive to boundary variation, the left and right sides of (1.2) are not comparable even if $G_1 = G \setminus \{p\}, p \in G$.

The distance ratio metric was introduced by F.W. Gehring and B.P. Palka [GP] and in the above, simplified, form by M. Vuorinen [Vu1] and it is frequently used in the study of hyperbolic type metrics [HIMPS] and geometric theory of functions. It is a basic fact that the above j-metric is closely related to the hyperbolic metric both for the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ and for the Poincaré half-space $\mathbb{H}^n$, [Vu2].

Quasi-invariance of j_G.

Given domains $G, G' \subset \mathbb{R}^n$ and an open continuous mapping $f : G \to G'$ with $fG \subset G'$ we consider the following condition: there exists a constant $C \ge 1$ such that for all $x, y \in G$ we have

$$j_{G'}(f(x), f(y)) \le C j_G(x,y)\,, \tag{1.3}$$

or, equivalently, that the mapping

$$f : (G, j_G) \to (G', j_{G'})$$

between metric spaces is Lipschitz continuous with the Lipschitz constant C.

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The hyperbolic metric in the unit ball or half space are Möbius invariant. However, the distance ratio metric j_G is not invariant under Möbius transformations. Therefore, it is natural to ask what the Lipschitz constants are for these metrics under conformal mappings or Möbius transformations in higher dimension. F. W. Gehring and B. G. Osgood proved that these metrics are not changed by more than a factor 2 under Möbius transformations, see [GO, proof of Theorem 4]:

Theorem 1.4. *If D and D' are proper subdomains of $\mathbb{R}^n$ and if f is a Möbius transformation of D onto D', then for all $x, y \in D$*

$$\frac{1}{2} j_D(x, y) \le j_{D'}(f(x), f(y)) \le 2 j_D(x, y).$$

It is easy to see that for a Möbius transformation $f : \mathbb{B}^n \to \mathbb{B}^n$ with $f(0) \neq 0$, and $x, y \in \mathbb{B}^n, x \neq y$, the $j_{\mathbb{B}^n}$ distances need not be the same. On the other hand, the next theorem from [SVW], conjectured in [KVZ], yields a sharp form of Theorem 1.4 for Möbius automorphisms of the unit ball.

Theorem 1.5. *A Möbius transformation $f : \mathbb{B}^n \to \mathbb{B}^n = f(\mathbb{B}^n)$ satisfies*

$$j_{\mathbb{B}^n}(f(x), f(y)) \le (1 + |f(0)|) j_{\mathbb{B}^n}(x, y)$$

for all $x, y \in \mathbb{B}^n$. The constant is best possible.

A similar result for a punctured disk was conjectured in [SVW]. The next theorem, our main result, settles this conjecture from [SVW] in the affirmative.

Theorem 1.6. *Let $a \in \mathbb{B}^2$ and $h : \mathbb{B}^2 \setminus \{0\} \to \mathbb{B}^2 \setminus \{a\}$ be a Möbius transformation with $h(0) = a$. Then for x, $y \in \mathbb{B}^2 \setminus \{0\}$*

$$j_{\mathbb{B}^2 \setminus \{a\}}(h(x), h(y)) \le C(a) j_{\mathbb{B}^2 \setminus \{0\}}(x, y),$$

where the constant $C(a) = 1 + (\log \frac{2+|a|}{2-|a|}) / \log 3$ is best possible.

Clearly the constant $C(a) < 1 + |a| < 2$ for all $a \in \mathbb{B}^2$ and hence the constant in Theorem 1.6 is smaller than the constant $1 + |f(0)|$ in Theorem 1.5 and far smaller than the constant 2 in Theorem 1.4.

If $a = 0$ in Theorem 1.6, then h is a rotation of the unit disk and hence a Euclidean isometry. Note that $C(0) = 1$, i.e. the result is sharp in this case.

The proof is based on Theorem 2.1 below and on Lemma 2.4, a monotone form of l'Hôpital's rule from [AVV, Theorem 1.25].

2. Preliminary results

In view of the definition of the distance ratio metric it is natural to expect that some properties of the logarithm will be needed. In the earlier paper [SVW], the classical Bernoulli inequality [Vu2, (3.6)] was applied for this purpose. Apparently now some other inequalities are needed and we use the following result, which is precise and allows us to get rid of logarithms in further calculations.

Theorem 2.1. *Let D and D' be proper subdomains of $\mathbb{R}^n$. For an open continuous mapping $f : D \to D'$ denote*

$$X = X(z,w) := \frac{|z-w|}{\min\{d_D(z), d_D(w)\}}; \;\; Y = Y(z,w) := \frac{|z-w|}{|f(z)-f(w)|}\frac{\min\{d_{D'}(f(z)), d_{D'}((f(w))\}}{\min\{d_D(z), d_D(w)\}}.$$

If there exists q, $0 \le q \le 1$ such that

$$q \le Y + \frac{Y-1}{X+1}, \tag{2.2}$$

then the inequality

$$j_{D'}(f(z), f(w)) \le \frac{2}{1+q} j_D(z,w),$$

holds for all $z, w \in D$.

Proof. The proof is based on the following assertion.

Lemma 2.3. *For $a \ge 0, q \in [0,1]$, we have*

$$\log\Big(\frac{q+e^a}{1+qe^a}\Big) \le \frac{1-q}{1+q}a.$$

Proof. Denote

$$f(a,q) := \log\Big(\frac{q+e^a}{1+qe^a}\Big) - \frac{1-q}{1+q}a.$$

By differentiation, we have

$$f'_a(a,q) = -\frac{q(1-q)}{1+q}\frac{(e^a-1)^2}{(1+qe^a)(q+e^a)},$$

we conclude that

$$f(a,q) \le f(0,q) = 0.$$

□

Now, since

$$X = \frac{|z-w|}{\min\{d_D(z), d_D(w)\}} = \exp(j_D(z,w)) - 1,$$

and

$$Y = \frac{|z-w|}{|f(z)-f(w)|}\frac{\min\{d_{D'}(f(z)), d_{D'}((f(w))\}}{\min\{d_D(z), d_D(w)\}} = \frac{\exp(j_D(z,w)) - 1}{\exp(j_{D'}(f(z), f(w))) - 1},$$

the condition (2.2) is equivalent to

$$\exp(j_{D'}(f(z), f(w))) \le \exp(j_D(z,w))\Big(\frac{q+e^{j_D(z,w)}}{1+qe^{j_D(z,w)}}\Big).$$

Therefore, by Lemma 2.3, we get

$$\begin{aligned} j_{D'}(f(z), f(w)) &\le j_D(z,w) + \log\Big(\frac{q+e^{j_D(z,w)}}{1+qe^{j_D(z,w)}}\Big) \\ &\le j_D(z,w) + \frac{1-q}{1+q}j_D(z,w) = \frac{2}{1+q}j_D(z,w). \end{aligned}$$

□

In the sequel we shall need the so-called *monotone form of l'Hôpital's rule.*

Lemma 2.4. [AVV, Theorem 1.25]. *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), and let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing(deceasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.4 has found numerous applications recently. See the bibliography of [AVZ] for a long list of applications to inequalities.

Lemma 2.5. *For positive numbers A, B, D and $0 < C < 1, \theta \geq 0$, we have*

1. The inequality

$$1+\frac{B}{D}\theta(1+\frac{D}{1+A})(1+\frac{B}{1-C}\theta) \leq (1+\frac{B}{D}\theta)(1+\frac{B}{1-C}\theta),$$

holds if and only if $B\theta \leq A+C$;

2. The function

$$\frac{\log(1+\frac{B}{1-C}\theta)}{\log(1+\frac{B}{D}\theta)}$$

is monotone increasing (decreasing) in θ if $C+D<1$ $(C+D>1)$.

Proof. Proof of the first part follows by direct calculation.

For the second part, set

$$f_1(\theta) = \log(1+\frac{B}{1-C}\theta), f_1(0)=0; \quad f_2(\theta) = \log(1+\frac{B}{D}\theta), f_2(0)=0.$$

Since

$$\frac{f_1'(\theta)}{f_2'(\theta)} = \frac{D+B\theta}{1-C+B\theta} = 1+\frac{C+D-1}{1-C+B\theta},$$

the proof follows according to Lemma 2.4. □

3. Proof of Theorem 1.6

3.1. Proof of Theorem 1.6.

For the proof, define $h(z) = \frac{z+a}{1+\bar{a}z}$ and suppose in the sequel that $|z| \geq |w|$. Then

$$j_G(z,w) = \log\Big(1+\frac{|z-w|}{\min\{|z|,|w|,1-|z|,1-|w|\}}\Big) = \log\Big(1+\frac{|z-w|}{\min\{|w|,1-|z|\}}\Big),$$

and

$$j_{G'}(h(z),h(w)) = \log\Big(1+\frac{|h(z)-h(w)|}{T}\Big),$$

where

$$T = T(a,z,w) := \min\{|h(z)-a|, |h(w)-a|, 1-|h(z)|, 1-|h(w)|\}.$$

In concert with the definition of the number T, the proof is divided into four cases. We shall consider each case separately applying Bernoulli inequality in the first case, its stronger form from Theorem 2.1 in the second one and a direct approach in the last two cases.

1. $T = |h(z) - a|$.

Since $|h(z) - a| = \frac{(1-|a|^2)|z|}{|1+\bar{a}z|}$ and $|h(z) - h(w)| = \frac{(1-|a|^2)|z-w|}{|1+\bar{a}z||1+\bar{a}w|}$, we have

$$j_{G'}(h(z), h(w)) = \log\Big(1 + \frac{|z-w|}{|z||1+\bar{a}w|}\Big).$$

Suppose firstly that $|w| \le 1 - |z|$. Since also $|w| \le 1 - |z| \le 1 - |w|$, we conclude that $0 \le |w| \le 1/2$. Hence, by the Bernoulli inequality (see e.g. [Vu2, (3.6)]), we get

$$j_{G'}(h(z), h(w)) \le \log\Big(1 + \frac{|z-w|}{|z|(1-|a||w|)}\Big) \le \log\Big(1 + \frac{|z-w|}{|w|(1-\frac{|a|}{2})}\Big)$$
$$\le \frac{1}{1-\frac{|a|}{2}} \log\Big(1 + \frac{|z-w|}{|w|}\Big) = \frac{1}{1-\frac{|a|}{2}} j_G(z, w).$$

Suppose now $1 - |z| \le |w| (\le |z|)$. Then $1/2 \le |z| < 1$.

Since in this case $(|z| - \frac{1}{2})(2 - |a|(1 + |z|)) \ge 0$, we easily obtain that

$$\frac{1}{|z|(1-|a||z|)} \le \frac{1}{(1-\frac{|a|}{2})(1-|z|)}.$$

Hence,

$$j_{G'}(h(z), h(w)) \le \log\Big(1 + \frac{|z-w|}{|z|(1-|a||w|)}\Big) \le \log\Big(1 + \frac{|z-w|}{|z|(1-|a||z|)}\Big)$$
$$\le \log\Big(1 + \frac{|z-w|}{(1-\frac{|a|}{2})(1-|z|)}\Big) \le \frac{1}{1-\frac{|a|}{2}} \log\Big(1 + \frac{|z-w|}{1-|z|}\Big) = \frac{1}{1-\frac{|a|}{2}} j_G(z, w).$$

2. $T = |h(w) - a|$.

This case can be treated by means of Theorem 2.1 with the same resulting constant $C_1(a) = \frac{2}{2-|a|}$.

Indeed, in terms of Theorem 2.1, we consider firstly the case $|w| \le 1 - |z|$.

We get

$$X = \frac{|z-w|}{|w|} \ge \frac{|z| - |w|}{|w|} = \frac{|z|}{|w|} - 1 = X^*,$$

and

$$Y = |1 + \bar{a}z| \ge 1 - |a||z| = Y^*.$$

Therefore,

$$Y + \frac{Y-1}{X+1} \ge Y^* - \frac{1-Y^*}{1+X^*} = 1 - |a||z| - |a|||z|\frac{|w|}{|z|}$$
$$= 1 - |a|(|w| + |z|) \ge 1 - |a| = q.$$

In the second case, i.e. when $1-|z|\le |w|$, we want to show that

$$Y+\frac{Y-1}{X+1}\ge 1-|a|\,,\ \text{with}\, X=\frac{|z-w|}{1-|z|}\,, Y=|1+\bar a z|\frac{|w|}{1-|z|}\,.$$

This is equivalent to

$$(Y-(1-|a|))(1+X)+Y\ge 1.$$

Since in this case

$$X=\frac{|z-w|}{1-|z|}\ge\frac{|z|-|w|}{1-|z|}:=X^*$$

and

$$Y=|1+\bar a z|\frac{|w|}{1-|z|}\ge(1-|a||z|)\frac{|w|}{1-|z|}=1-|a||z|+(|w|+|z|-1)\frac{1-|a||z|}{1-|z|}:=Y^*,$$

we get

$$\begin{aligned}&(Y-(1-|a|))(1+X)+Y-1\ge(Y^*-(1-|a|))(1+X^*)+Y^*-1\\ =&\Big[|a|(1-|z|)+(|w|+|z|-1)\frac{1-|a||z|}{1-|z|}\Big]\frac{1-|w|}{1-|z|}-|a||z|+(|w|+|z|-1)\frac{1-|a||z|}{1-|z|}\\ &\ge |a|(1-|w|-|z|)+(|w|+|z|-1)\frac{1-|a||z|}{1-|z|}=(|w|+|z|-1)\frac{1-|a|}{1-|z|}\ge 0.\end{aligned}$$

Therefore by Theorem 2.1, in both cases we get

$$j_{G'}(h(z),h(w))\le\frac{2}{1+q}j_G(z,w)=\frac{2}{2-|a|}j_G(z,w)=C_1(a)j_G(z,w).$$

3. $T=1-|h(z)|$.

In this case, applying well-known assertions

$$|1+\bar a z|^2-|a+z|^2=(1-|a|^2)(1-|z|^2);\quad |h(z)|\le\frac{|a|+|z|}{1+|a||z|},$$

and

$$|1+\bar a w|\ge 1-|a||w|(\ge 1-|a||z|),$$

we get

$$\begin{aligned}j_{G'}(h(z),h(w))&=\log\Big(1+\frac{|z-w|(1-|a|^2)}{|1+\bar a w|(|1+\bar a z|-|z+a|)}\Big)=\log\Big(1+\frac{|z-w|(|1+\bar a z|+|z+a|)}{|1+\bar a w|(1-|z|^2)}\Big)\\ &=\log\Big(1+\frac{|z-w|}{1-|z|^2}|1+\frac{\bar a(z-w)}{1+\bar a w}|(1+\frac{|z+a|}{|1+\bar a z|})\Big)\le\log\Big(1+\frac{|z-w|}{1-|z|}(1+\frac{|a||z-w|}{1-|a||w|})(1+\frac{|a|(1-|z|)}{1+|a||z|})\Big).\end{aligned}$$

Applying here Lemma 2.5, part 1., with

$$A=|a||z|,\ B=|a|,\ C=|a||w|, D=|a|(1-|z|),\ \theta=|z-w|,$$

we obtain

$$j_{G'}(h(z),h(w))\le\log\Big[\Big(1+\frac{|z-w|}{1-|z|}\Big)\Big(1+\frac{|a||z-w|}{1-|a||w|}\Big)\Big]. \tag{3.2}$$

Suppose that $1-|z| \le |w| (\le |z|)$. By Lemma 2.5, part 2., with

$$B=|a|,\ C=|a||z|,\ D=|a|(1-|z|),\ \theta=|z-w|,$$

we get

$$J(z,w;a) := \frac{j_{G'}(h(z),h(w))}{j_G(z,w)} \le 1+\frac{\log(1+\frac{|a||z-w|}{1-|a||z|})}{\log\left(1+\frac{|z-w|}{1-|z|}\right)}$$

$$\le 1+\frac{\log(1+\frac{2|a||z|}{1-|a||z|})}{\log\left(1+\frac{2|z|}{1-|z|}\right)},$$

because in this case we have $C+D=|a|<1$ and $|z-w| \le 2|z|$.

Since the last function is monotone decreasing in $|z|$ and $|z| \ge 1/2$, we obtain

$$J(z,w;a) \le 1+\frac{\log(\frac{1+\frac{1}{2}|a|}{1-\frac{1}{2}|a|})}{\log\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)} = 1+(\log\frac{2+|a|}{2-|a|})/\log 3 := C_2(a).$$

Let now $|w| \le 1-|z| (\le 1-|w|)$. The estimation (3.2) and Lemma 2.5, part 2., with

$$B=|a|,\ C=D=|a||w|,\ \theta=|z-w|,$$

yield

$$J(z,w;a) \le \frac{\log\left[\left(1+\frac{|z-w|}{1-|z|}\right)\left(1+\frac{|a||z-w|}{1-|a||w|}\right)\right]}{\log\left(1+\frac{|z-w|}{|w|}\right)} \le \frac{\log\left[\left(1+\frac{|z-w|}{|w|}\right)\left(1+\frac{|a||z-w|}{1-|a||w|}\right)\right]}{\log\left(1+\frac{|z-w|}{|w|}\right)}$$

$$=1+\frac{\log\left(1+\frac{|a||z-w|}{1-|a||w|}\right)}{\log\left(1+\frac{|z-w|}{|w|}\right)} \le 1+\frac{\log\left(1+\frac{|a|}{1-|a||w|}\right)}{\log\left(1+\frac{1}{|w|}\right)},$$

since $C+D=2|a||w| \le |a| < 1$ and $0 \le |z-w| \le |z|+|w| \le 1$.

Denote the last function as $g(|w|)$ and let $|w|=r,\ 0<r\le 1/2$. Since

$$g'(r)=\frac{|a|^2}{(1-r|a|)(1+(1-r)|a|)\log(1+1/r)}+\frac{\log\left(1+\frac{|a|}{1-|a|r}\right)}{r(1+r)\log^2(1+1/r)}>0,$$

we finally obtain

$$J(z,w;a) \le 1+\frac{\log\left(1+\frac{|a|}{1-|a|/2}\right)}{\log(1+2)} = C_2(a).$$

4. $T=1-|h(w)|$.

This case can be considered analogously with the previous one.

$$j_{G'}(h(z),h(w)) = \log\left(1+\frac{|z-w|(1-|a|^2)}{|1+\bar{a}z|(|1+\bar{a}w|-|w+a|)}\right) = \log\left(1+\frac{|z-w|(|1+\bar{a}w|+|w+a|)}{|1+\bar{a}z|(1-|w|^2)}\right)$$

$$= \log\Big(1+\frac{|z-w|}{1-|w|^2}|1+\frac{\bar a(w-z)}{1+\bar a z}|(1+\frac{|w+a|}{|1+\bar a w|})\Big) \le \log\Big(1+\frac{|z-w|}{1-|w|}(1+\frac{|a||z-w|}{1-|a||z|})(1+\frac{|a|(1-|w|)}{1+|a||w|})\Big)$$

Applying Lemma 2.5, part 1., with

$$A=|a||w|,\ B=|a|,\ C=|a||z|, D=|a|(1-|w|),\ \theta=|z-w|,$$

we obtain

$$j_{G'}(h(z),h(w)) \le \log\Big[\Big(1+\frac{|z-w|}{1-|w|}\Big)\Big(1+\frac{|a||z-w|}{1-|a||z|}\Big)\Big]. \tag{3.3}$$

Suppose that $1-|z|\le |w| (\le |z|)$. We get

$$J(z,w;a) := \frac{j_{G'}(h(z),h(w))}{j_G(z,w)} \le \frac{\log\Big[\Big(1+\frac{|z-w|}{1-|z|}\Big)\Big(1+\frac{|a||z-w|}{1-|a||z|}\Big)\Big]}{\log\Big(1+\frac{|z-w|}{1-|z|}\Big)}$$

$$= 1+\frac{\log(1+\frac{|a||z-w|}{1-|a||z|})}{\log\Big(1+\frac{|z-w|}{1-|z|}\Big)}$$

and this inequality is already considered above.

In the case $|w|\le 1-|z|\le 1-|w|$, we have

$$J(z,w;a) \le \frac{\log\Big[\Big(1+\frac{|z-w|}{1-|w|}\Big)\Big(1+\frac{|a||z-w|}{1-|a||z|}\Big)\Big]}{\log\Big(1+\frac{|z-w|}{|w|}\Big)} \le \frac{\log\Big[\Big(1+\frac{|z-w|}{|w|}\Big)\Big(1+\frac{|a||z-w|}{1-|a|(1-|w|)}\Big)\Big]}{\log\Big(1+\frac{|z-w|}{|w|}\Big)}$$

$$= 1+\frac{\log\Big(1+\frac{|a||z-w|}{1-|a|(1-|w|)}\Big)}{\log\Big(1+\frac{|z-w|}{|w|}\Big)} \le 1+\frac{\log\Big(1+\frac{|a|}{1-|a|(1-|w|)}\Big)}{\log\Big(1+\frac{1}{|w|}\Big)},$$

where the last inequality follows from Lemma 2.5, part 2., with

$$B=|a|,\ C=|a|(1-|w|),\ D=|a||w|,\ \theta=|z-w|,$$

since $C+D=|a|<1$ and $|z-w|\le |z|+|w|\le 1$.

Denote now $|w|=r$ and let $k(r)=k_1(r)/k_2(r)$ with

$$k_1(r)=\log\Big(1+\frac{|a|}{1-|a|(1-r)}\Big);\ k_2(r)=\log\Big(1+\frac{1}{r}\Big).$$

We shall show now that the function $k(r)$ is monotone increasing on the positive part of real axis.

Indeed, since $k_1(\infty)=k_2(\infty)=0$ and

$$k_1'(r)/k_2'(r) = \frac{|a|^2 r(1+r)}{(1+|a|r)(1-|a|+|a|r)} = \frac{|a|(1+r)}{1+|a|r}\frac{|a|r}{1-|a|+|a|r}$$

$$= (1-\frac{1-|a|}{1+|a|r})(1-\frac{1-|a|}{1-|a|+|a|r}),$$

with both functions in parenthesis evidently increasing on $\mathbb{R}^+$, the conclusion follows from Lemma 2.4.

Since in this case $0<r\le 1/2$, we also obtain that

$$J(z,w;a) \le 1 + \frac{\log\Big(1+\frac{|a|}{1-|a|/2}\Big)}{\log\Big(1+2\Big)} = C_2(a).$$

The constant $C_2(a)$ is sharp since $J(\frac{a}{2|a|},\frac{-a}{2|a|};a) = C_2(a)$.

Because $C_2(a) = 1+(\log\frac{2+a}{2-a})/\log 3 > \frac{1}{1-\frac{|a|}{2}} = C_1(a)$, we conclude that the best possible upper bound C is $C = C_2(a)$ $\square$.

Finally, in order to widen the topic started with Conjecture 1, we consider the following: Let $h : \mathbb{D} \to \mathbb{D}$ be Möbius map with $h(0) = a$. A challenging problem is to determine best possible j-Lip constants $C(m,a)$ such that

$$j(h^m(z), h^m(w)) \le C(m,a) j(z,w),$$

for all $z, w \in \mathbb{D}$ and $m \in \mathbb{N}$.

It is not difficult to show that $C(m,a) \le 1+|a| = C(1,a),\ m \in \mathbb{N}$. Therefore, the following question naturally arise.

Q1. *Is the sequence $C(m,a)$ monotone decreasing in m?*

A partial answer is given in the next

Theorem 3.4. *The sequence $C(2^n,a), n \in \mathbb{N}$ is monotone decreasing in n.*

Proof. Indeed, since

$$\frac{|h^{2^{n+1}}(z)-h^{2^{n+1}}(w)|}{1-\max\{|h(z)|^{2^{n+1}},|h(w)|^{2^{n+1}}\}} = \frac{|h^{2^n}(z)+h^{2^n}(w)|}{1+\max\{|h(z)|^{2^n},|h(w)|^{2^n}\}}\frac{|h^{2^n}(z)-h^{2^n}(w)|}{1-\max\{|h(z)|^{2^n},|h(w)|^{2^n}\}}$$
$$\le \frac{2\max\{|h|^{2^n}(z),|h|^{2^n}(w)\}}{1+\max\{|h(z)|^{2^n},|h(w)|^{2^n}\}}\frac{|h^{2^n}(z)-h^{2^n}(w)|}{1-\max\{|h(z)|^{2^n},|h(w)|^{2^n}\}} \le \frac{|h^{2^n}(z)-h^{2^n}(w)|}{1-\max\{|h(z)|^{2^n},|h(w)|^{2^n}\}},$$

we conclude that

$$j(h^{2^{n+1}}(z),h^{2^{n+1}}(w)) \le j(h^{2^n}(z),h^{2^n}(w)),$$

i.e.,

$$C(2^{n+1},a) = \sup_{z,w\in\mathbb{D}}\frac{j(h^{2^{n+1}}(z),h^{2^{n+1}}(w))}{j(z,w)} \le \sup_{z,w\in\mathbb{D}}\frac{j(h^{2^n}(z),h^{2^n}(w))}{j(z,w)} = C(2^n,a).$$

$\square$

Q2. *Is it true that $C(m,(m+1)^{-1}) = 1$ for $m \ge 2$?*

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